

REAL HYPERSURFACES IN THE COMPLEX QUADRIC WITH COMMUTING RICCI TENSOR

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ABSTRACT. We introduce the notion of commuting Ricci tensor for real hypersurfaces in the complex quadric $Q^m = SO_{m+2}/SO_m SO_2$. It is shown that the commuting Ricci tensor gives that the unit normal vector field N becomes \mathfrak{A} -principal or \mathfrak{A} -isotropic. Then according to each case, we give a complete classification of real hypersurfaces in $Q^m = SO_{m+2}/SO_m SO_2$ with commuting Ricci tensor.

1. INTRODUCTION

In a class of Hermitian symmetric spaces of rank 2, usually we can give examples of Riemannian symmetric spaces $SU_{m+2}/S(U_2 U_m)$ and $SU_{2,m}/S(U_2 U_m)$, which are said to be complex two-plane Grassmannians and complex hyperbolic two-plane Grassmannians respectively (see [12], [13], [14], [15] and [17]). These are viewed as Hermitian symmetric spaces and quaternionic Kähler symmetric spaces equipped with the Kähler structure J and the quaternionic Kähler structure \mathfrak{J} and they have rank 2.

Among the other different type of Hermitian symmetric space with rank 2 in the class of compact type, we can give the example of complex quadric $Q^m = SO_{m+2}/SO_m SO_2$, which is a complex hypersurface in complex projective space $\mathbb{C}P^{m+1}$ (see Suh [16], [18], and Smyth [11]). The complex quadric can also be regarded as a kind of real Grassmann manifolds of compact type with rank 2 (see Kobayashi and Nomizu [5]). Accordingly, the complex quadric admits two important geometric structures, a complex conjugation structure A and a Kähler structure J , which anti-commute with each other, that is, $AJ = -JA$. Then for $m \geq 2$ the triple (Q^m, J, g) is a Hermitian symmetric space of compact type with rank 2 and its maximal sectional curvature is equal to 4 (see Klein [2] and Reckziegel [10]).

In the complex projective space $\mathbb{C}P^{m+1}$ and the quaternionic projective space $\mathbb{Q}P^{m+1}$ some classifications related to commuting Ricci tensor or commuting structure Jacobi operator were investigated by Kimura [3], [4], Pérez [6] and Pérez and Suh [7], [8] respectively. Under the invariance of the shape operator along some distributions a new classification in the complex 2-plane Grassmannian $G_2(\mathbb{C}^{m+2}) = SU_{m+2}/S(U_m U_2)$ was investigated. By using this classification Pérez and Suh [9] proved a non-existence property for Hopf hypersurfaces in $G_2(\mathbb{C}^{m+2})$ with parallel and commuting Ricci tensor. Recently, Hwang, Lee and Woo [1] considered the notion of semi-parallel symmetric operators and

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obtained a complete classification for Hopf hypersurfaces in $G_2(\mathbb{C}^{m+2})$. Moreover, Suh [12] strengthened this result to hypersurfaces in $G_2(\mathbb{C}^{m+2})$ with commuting Ricci tensor and gave a characterization of real hypersurfaces in $G_2(\mathbb{C}^{m+2}) = SU_{m+2}/S(U_m U_2)$ as follows:

Theorem A. *Let M be a Hopf real hypersurface in $G_2(\mathbb{C}^{m+2})$ with commuting Ricci tensor, $m \geq 3$. Then M is locally congruent to a tube of radius r over a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$.*

Moreover, Suh [17] studied another classification for Hopf hypersurfaces in complex hyperbolic two-plane Grassmannians $SU_{2,m}/S(U_2 U_m)$ with commuting Ricci tensor as follows:

Theorem B. *Let M be a Hopf hypersurface in $SU_{2,m}/S(U_2 \cdot U_m)$ with commuting Ricci tensor, $m \geq 3$. Then M is locally congruent to an open part of a tube around some totally geodesic $SU_{2,m-1}/S(U_2 \cdot U_{m-1})$ in $SU_{2,m}/S(U_2 \cdot U_m)$ or a horosphere whose center at infinity with $JX \in \mathfrak{J}X$ is singular.*

It is known that the Reeb flow on a real hypersurface in $G_2(\mathbb{C}^{m+2})$ is isometric if and only if M is an open part of a tube around a totally geodesic $G_2(\mathbb{C}^{m+1}) \subset G_2(\mathbb{C}^{m+2})$. Corresponding to this result, in [15] we asserted that the Reeb flow on a real hypersurface in $SU_{2,m}/S(U_2 U_m)$ is isometric if and only if M is an open part of a tube around a totally geodesic $SU_{2,m-1}/S(U_2 U_{m-1}) \subset SU_{2,m}/S(U_2 U_m)$. Here, the Reeb flow on real hypersurfaces in $SU_{m+2}/S(U_m U_2)$ or $SU_{2,m}/S(U_2 U_m)$ is said to be *isometric* if the shape operator commutes with the structure tensor. In papers [16] and [18] due to Suh, we have introduced this problem for real hypersurfaces in the complex quadric $Q^m = SO_{m+2}/SO_m SO_2$ and obtained the following result:

Theorem C. *Let M be a real hypersurface of the complex quadric Q^m , $m \geq 3$. The Reeb flow on M is isometric if and only if m is even, say $m = 2k$, and M is an open part of a tube around a totally geodesic $\mathbb{C}P^k \subset Q^{2k}$.*

Now at each point $z \in M$ let us consider a maximal \mathfrak{A} -invariant subspace \mathcal{Q}_z of $T_z M$, $z \in M$, defined by

$$\mathcal{Q}_z = \{X \in T_z M \mid AX \in T_z M \text{ for all } A \in \mathfrak{A}_z\}$$

of $T_z M$, $z \in M$. Thus for a case where the unit normal vector field N is \mathfrak{A} -isotropic it can be easily checked that the orthogonal complement $\mathcal{Q}_z^\perp = \mathcal{C}_z \ominus \mathcal{Q}_z$, $z \in M$, of the distribution \mathcal{Q} in the complex subbundle \mathcal{C} , becomes $\mathcal{Q}_z^\perp = \text{Span}\{A\xi, AN\}$. Here it can be easily checked that the vector fields $A\xi$ and AN belong to the tangent space $T_z M$, $z \in M$ if the unit normal vector field N becomes \mathfrak{A} -isotropic. Thus for a case where the unit normal vector field N is \mathfrak{A} -isotropic it can be easily checked that the orthogonal complement $\mathcal{Q}_z^\perp = \mathcal{C}_z \ominus \mathcal{Q}_z$, $z \in M$, of the distribution \mathcal{Q} in the complex subbundle \mathcal{C} , becomes $\mathcal{Q}_z^\perp = \text{Span}\{A\xi, AN\}$. Moreover, the vector fields $A\xi$ and AN belong to the tangent space $T_z M$, $z \in M$ if the unit normal vector field N becomes \mathfrak{A} -isotropic. Then motivated by the above result, in [18] we gave another theorem for real hypersurfaces in the complex quadric Q^m with parallel Ricci tensor and \mathfrak{A} -isotropic unit normal.

Apart from the complex structure J there is another distinguished geometric structure on Q^m , namely a parallel rank two vector bundle \mathfrak{A} which contains an S^1 -bundle of real

structures, that is, complex conjugations A on the tangent spaces of Q^m . This geometric structure determines a maximal \mathfrak{A} -invariant subbundle \mathcal{Q} of the tangent bundle TM of a real hypersurface M in Q^m .

When we consider a hypersurface M in the complex quadric Q^m , under the assumption of some geometric properties the unit normal vector field N of M in Q^m can be divided into two classes if either N is \mathfrak{A} -isotropic or \mathfrak{A} -principal (see [16], and [18]). In the first case where N is \mathfrak{A} -isotropic, it was known that M is locally congruent to a tube over a totally geodesic $\mathbb{C}P^k$ in Q^{2k} . In the second case, when the unit normal N is \mathfrak{A} -principal, we proved that a contact hypersurface M in Q^m is locally congruent to a tube over a totally geodesic and totally real submanifold S^m in Q^m (see [18]).

In the study of complex two-plane Grassmannian $G_2(\mathbb{C}^{m+2})$ or complex hyperbolic two-plane Grassmannian $SU_{2,m}/S(U_2 \cdot U_m)$ we studied hypersurfaces with parallel Ricci tensor and gave non-existence properties respectively (see [13] and [20]). In [18] we also considered the notion of parallel Ricci tensor $\nabla \text{Ric} = 0$ for hypersurfaces M in Q^m . As a generalization of such facts, we consider the notion of harmonic curvature, that is, $(\nabla_X \text{Ric})Y = (\nabla_Y \text{Ric})X$ for any tangent vector fields X and Y on M in Q^m and proved the following (see [19])

Theorem D. *Let M be a Hopf real hypersurface in the complex quadric Q^m , $m \geq 4$, with harmonic curvature and \mathfrak{A} -isotropic unit normal N . If the shape operator commutes with the structure tensor on the distribution \mathcal{Q}^\perp , then M is locally congruent to an open part of a tube around k -dimensional complex projective space $\mathbb{C}P^k$ in Q^m , $m = 2k$, or M has at most 6 distinct constant principal curvatures given by*

$$\alpha, \quad \gamma = 0(\alpha), \quad \lambda_1, \quad \mu_1, \quad \lambda_2 \quad \text{and} \quad \mu_2$$

with corresponding principal curvature spaces

$$T_\alpha = [\xi], \quad T_\gamma = [A\xi, AN], \quad \phi(T_{\lambda_1}) = T_{\mu_1}, \quad \phi T_{\lambda_2} = T_{\mu_2}.$$

$$\dim T_{\lambda_1} + \dim T_{\lambda_2} = m - 2, \quad \dim T_{\mu_1} + \dim T_{\mu_2} = m - 2.$$

Here four roots λ_i and μ_i , $i = 1, 2$ satisfy the equation

$$2x^2 - 2\beta x + 2 + \alpha\beta = 0,$$

where the function β denotes $\beta = \frac{\alpha^2 + 2 \pm \sqrt{(\alpha^2 + 2)^2 + 4\alpha h}}{\alpha}$. In particular, $\alpha = \sqrt{\frac{2m-1}{2}}$, $\gamma (= \alpha) = \sqrt{\frac{2m-1}{2}}$, $\lambda = 0$, $\mu = -\frac{2\sqrt{2}}{\sqrt{2m-1}}$, with multiplicities 1, 2, $m-2$ and $m-2$ respectively.

But from the assumption of harmonic curvature, it was impossible to derive the fact that either the unit normal N is \mathfrak{A} -isotropic or \mathfrak{A} -principal. So in [19] we gave a complete classification with the further assumption of \mathfrak{A} -isotropic as in Theorem D. For the case where the unit normal vector field N is \mathfrak{A} -principal we have proved that real hypersurfaces in Q^m with harmonic curvature can not be existed.

But fortunately when we consider Ricci commuting, that is, $\text{Ric} \cdot \phi = \phi \cdot \text{Ric}$ for hypersurfaces M in Q^m , we can assert that the unit normal vector field N becomes either \mathfrak{A} -isotropic or \mathfrak{A} -principal. Then motivated by such a result and using Theorem C, in this paper we give a complete classification for real hypersurfaces in the complex quadric Q^m with commuting Ricci tensor, that is, $\text{Ric} \cdot \phi = \phi \cdot \text{Ric}$ as follows:

Main Theorem. *Let M be a Hopf real hypersurface in the complex quadric Q^m , $m \geq 4$, with commuting Ricci tensor. If the shape operator commutes with the structure tensor on the distribution \mathcal{Q}^\perp , then M is locally congruent to an open part of a tube around totally geodesic $\mathbb{C}P^k$ in Q^{2k} , $m = 2k$ or M has 3 distinct constant principal curvatures given by*

$$\alpha = \sqrt{2(m-3)}, \gamma = 0, \lambda = 0, \text{ and } \mu = -\frac{2}{\sqrt{2(m-3)}} \text{ or}$$

$$\alpha = \sqrt{\frac{2}{3}(m-3)}, \gamma = 0, \lambda = 0, \text{ and } \mu = -\frac{\sqrt{6}}{\sqrt{m-3}}$$

with corresponding principal curvature spaces respectively

$$T_\alpha = [\xi], T_\gamma = [A\xi, AN], \phi(T_\lambda) = T_\mu, \text{ and } \dim T_\lambda = \dim T_\mu = m - 2.$$

Remark 1.1. *In the main theorem the second and the third ones can be explained geometrically as follows: the real hypersurface M is locally congruent to $M_1 \times \mathbb{C}$, where M_1 is a tube of radius $r = \frac{1}{\sqrt{2}} \tan^{-1} \sqrt{m-3}$ or respectively, of radius $r = \frac{1}{\sqrt{2}} \tan^{-1} \sqrt{\frac{m-3}{3}}$, around $(m-1)$ -dimensional sphere S^{m-1} in Q^{m-1} . That is, M_1 is a contact hypersurface defined by $S\phi + \phi S = k\phi$, $k = -\frac{2}{\sqrt{2(m-3)}}$, and $k = -\frac{\sqrt{6}}{\sqrt{m-3}}$ respectively (see Suh [18]). By the Segre embedding, the embedding $M_1 \times \mathbb{C} \subset Q^{m-1} \times \mathbb{C} \subset Q^m$ is defined by $(z_0, z_1, \dots, z_m, w) \rightarrow (z_0 w, z_1 w, \dots, z_m w, 0)$. Here $(z_0 w)^2 + (z_1 w)^2 + \dots + (z_m w)^2 = (z_0^2 + \dots + z_m^2)w^2 = 0$, where $\{z_0, \dots, z_m\}$ denotes a coordinate system in Q^{m-1} satisfying $z_0^2 + \dots + z_m^2 = 0$.*

Our paper is organized as follows. In section 2 we present basic material about the complex quadric Q^m , including its Riemannian curvature tensor and a description of its singular vectors of Q^m like \mathfrak{A} -principal or \mathfrak{A} -isotropic unit normal vector field. In section 3, we investigate the geometry of this subbundle \mathcal{Q} for hypersurfaces in Q^m and some equations including Codazzi and fundamental formulas related to the vector fields ξ , N , $A\xi$, and AN for the complex conjugation A of M in Q^m .

In section 4, the first step is to derive the formula of Ricci commuting from the equation of Gauss for real hypersurfaces M in Q^m and to get a key lemma that the unit normal vector field N can be divided into two classes of normal vector such that N is either \mathfrak{A} -isotropic or \mathfrak{A} -principal, and show that a real hypersurface in Q^m , $m = 2k$, which is a tube over a totally geodesic $\mathbb{C}P^k$ in Q^{2k} naturally admits a commuting Ricci tensor. In sections 5, by the expressions of the shape operator S for real hypersurfaces M in Q^m , we present the proof of Main Theorem with \mathfrak{A} -isotropic unit normal.

In section 6, we give a complete proof of Main Theorem with \mathfrak{A} -principal unit normal. The first part of this proof is devoted to give some fundamental formulas from Ricci commuting and \mathfrak{A} -principal unit normal vector field. Then in the latter part of the proof we will use the decomposition of two eigenspaces of the complex conjugation A in Q^m such that $T_z M = V(A) \oplus JV(A)$, where two eigenspaces are defined by $V(A) = \{X \in T_z Q^m \mid AX = X\}$ and $JV(A) = \{X \in T_z Q^m \mid AX = -X\}$ respectively.

2. THE COMPLEX QUADRIC

For more background to this section we refer to [2], [5], [10], [16], and [18]. The complex quadric Q^m is the complex hypersurface in $\mathbb{C}P^{m+1}$ which is defined by the equation $z_0^2 + \dots + z_{m+1}^2 = 0$, where z_0, \dots, z_{m+1} are homogeneous coordinates on $\mathbb{C}P^{m+1}$. We equip Q^m with the Riemannian metric g which is induced from the Fubini-Study metric \bar{g} on $\mathbb{C}P^{m+1}$ with constant holomorphic sectional curvature 4. The Fubini-Study metric \bar{g} is defined by $\bar{g}(X, Y) = \Phi(JX, Y)$ for any vector fields X and Y on $\mathbb{C}P^{m+1}$ and a globally closed $(1, 1)$ -form Φ given by $\Phi = -4i\partial\bar{\partial}\log f_j$ on an open set $U_j = \{[z^0, z^1, \dots, z^{m+1}] \in \mathbb{C}P^{m+1} | z^j \neq 0\}$, where the function f_j denotes $f_j = \sum_{k=0}^{m+1} t_j^k \bar{t}_j^k$, and $t_j^k = \frac{z^k}{z^j}$ for $j, k = 0, \dots, m+1$. Then naturally the Kähler structure on $\mathbb{C}P^{m+1}$ induces canonically a Kähler structure (J, g) on the complex quadric Q^m .

The complex projective space $\mathbb{C}P^{m+1}$ is a Hermitian symmetric space of the special unitary group SU_{m+2} , namely $\mathbb{C}P^{m+1} = SU_{m+2}/S(U_{m+1}U_1)$. We denote by $o = [0, \dots, 0, 1] \in \mathbb{C}P^{m+1}$ the fixed point of the action of the stabilizer $S(U_{m+1}U_1)$. The special orthogonal group $SO_{m+2} \subset SU_{m+2}$ acts on $\mathbb{C}P^{m+1}$ with cohomogeneity one. The orbit containing o is a totally geodesic real projective space $\mathbb{R}P^{m+1} \subset \mathbb{C}P^{m+1}$. The second singular orbit of this action is the complex quadric $Q^m = SO_{m+2}/SO_m SO_2$. This homogeneous space model leads to the geometric interpretation of the complex quadric Q^m as the Grassmann manifold $G_2^+(\mathbb{R}^{m+2})$ of oriented 2-planes in \mathbb{R}^{m+2} . It also gives a model of Q^m as a Hermitian symmetric space of rank 2. The complex quadric Q^1 is isometric to a sphere S^2 with constant curvature, and Q^2 is isometric to the Riemannian product of two 2-spheres with constant curvature. For this reason we will assume $m \geq 3$ from now on.

For a nonzero vector $z \in \mathbb{C}^{m+2}$ we denote by $[z]$ the complex span of z , that is, $[z] = \{\lambda z | \lambda \in \mathbb{C}\}$. Note that by definition $[z]$ is a point in $\mathbb{C}P^{m+1}$. As usual, for each $[z] \in \mathbb{C}P^{m+1}$ we identify $T_{[z]}\mathbb{C}P^{m+1}$ with the orthogonal complement $\mathbb{C}^{m+2} \ominus [z]$ of $[z]$ in \mathbb{C}^{m+2} . For $[z] \in Q^m$ the tangent space $T_{[z]}Q^m$ can then be identified canonically with the orthogonal complement $\mathbb{C}^{m+2} \ominus ([z] \oplus [\bar{z}])$ of $[z] \oplus [\bar{z}]$ in \mathbb{C}^{m+2} (see Kobayashi and Nomizu [5]). Note that $\bar{z} \in \nu_{[z]}Q^m$ is a unit normal vector of Q^m in $\mathbb{C}P^{m+1}$ at the point $[z]$.

We denote by $A_{\bar{z}}$ the shape operator of Q^m in $\mathbb{C}P^{m+1}$ with respect to the unit normal \bar{z} . It is defined by $A_{\bar{z}}w = \bar{\nabla}_w \bar{z} = \bar{w}$ for a complex Euclidean connection $\bar{\nabla}$ induced from \mathbb{C}^{m+2} and all $w \in T_{[z]}Q^m$. That is, the shape operator $A_{\bar{z}}$ is just a complex conjugation restricted to $T_{[z]}Q^m$. Moreover, it satisfies the following for any $w \in T_{[z]}Q^m$ and any $\lambda \in S^1 \subset \mathbb{C}$

$$\begin{aligned} A_{\lambda\bar{z}}^2 w &= A_{\lambda\bar{z}} A_{\lambda\bar{z}} w = A_{\lambda\bar{z}} \lambda \bar{w} \\ &= \lambda A_{\bar{z}} \lambda \bar{w} = \lambda \bar{\nabla}_{\lambda\bar{w}} \bar{z} = \lambda \bar{\lambda} \bar{w} \\ &= |\lambda|^2 w = w. \end{aligned}$$

Accordingly, $A_{\lambda\bar{z}}^2 = I$ for any $\lambda \in S^1$. So the shape operator $A_{\bar{z}}$ becomes an anti-commuting involution such that $A_{\bar{z}}^2 = I$ and $AJ = -JA$ on the complex vector space $T_{[z]}Q^m$ and

$$T_{[z]}Q^m = V(A_{\bar{z}}) \oplus JV(A_{\bar{z}}),$$

where $V(A_{\bar{z}}) = \mathbb{R}^{m+2} \cap T_{[z]}Q^m$ is the $(+1)$ -eigenspace and $JV(A_{\bar{z}}) = i\mathbb{R}^{m+2} \cap T_{[z]}Q^m$ is the (-1) -eigenspace of $A_{\bar{z}}$. That is, $A_{\bar{z}}X = X$ and $A_{\bar{z}}JX = -JX$, respectively, for any $X \in V(A_{\bar{z}})$.

Geometrically this means that the shape operator $A_{\bar{z}}$ defines a real structure on the complex vector space $T_{[z]}Q^m$, or equivalently, is a complex conjugation on $T_{[z]}Q^m$. Since the real codimension of Q^m in $\mathbb{C}P^{m+1}$ is 2, this induces an S^1 -subbundle \mathfrak{A} of the endomorphism bundle $\text{End}(TQ^m)$ consisting of complex conjugations.

There is a geometric interpretation of these conjugations. The complex quadric Q^m can be viewed as the complexification of the m -dimensional sphere S^m . Through each point $[z] \in Q^m$ there exists a one-parameter family of real forms of Q^m which are isometric to the sphere S^m . These real forms are congruent to each other under action of the center SO_2 of the isotropy subgroup of SO_{m+2} at $[z]$. The isometric reflection of Q^m in such a real form S^m is an isometry, and the differential at $[z]$ of such a reflection is a conjugation on $T_{[z]}Q^m$. In this way the family \mathfrak{A} of conjugations on $T_{[z]}Q^m$ corresponds to the family of real forms S^m of Q^m containing $[z]$, and the subspaces $V(A) \subset T_{[z]}Q^m$ correspond to the tangent spaces $T_{[z]}S^m$ of the real forms S^m of Q^m .

The Gauss equation for $Q^m \subset \mathbb{C}P^{m+1}$ implies that the Riemannian curvature tensor \bar{R} of Q^m can be described in terms of the complex structure J and the complex conjugations $A \in \mathfrak{A}$:

$$\begin{aligned} \bar{R}(X, Y)Z &= g(Y, Z)X - g(X, Z)Y + g(JY, Z)JX - g(JX, Z)JY - 2g(JX, Y)JZ \\ &\quad + g(AY, Z)AX - g(AX, Z)AY + g(JAY, Z)JAX - g(JAX, Z)JAY. \end{aligned}$$

Note that J and each complex conjugation A anti-commute, that is, $AJ = -JA$ for each $A \in \mathfrak{A}$.

Recall that a nonzero tangent vector $W \in T_{[z]}Q^m$ is called singular if it is tangent to more than one maximal flat in Q^m . There are two types of singular tangent vectors for the complex quadric Q^m :

1. If there exists a conjugation $A \in \mathfrak{A}$ such that $W \in V(A)$, then W is singular. Such a singular tangent vector is called \mathfrak{A} -principal.
2. If there exist a conjugation $A \in \mathfrak{A}$ and orthonormal vectors $X, Y \in V(A)$ such that $W/\|W\| = (X + JY)/\sqrt{2}$, then W is singular. Such a singular tangent vector is called \mathfrak{A} -isotropic.

For every unit tangent vector $W \in T_{[z]}Q^m$ there exist a conjugation $A \in \mathfrak{A}$ and orthonormal vectors $X, Y \in V(A)$ such that

$$W = \cos(t)X + \sin(t)JY$$

for some $t \in [0, \pi/4]$. The singular tangent vectors correspond to the values $t = 0$ and $t = \pi/4$. If $0 < t < \pi/4$ then the unique maximal flat containing W is $\mathbb{R}X \oplus \mathbb{R}JY$. Later we will need the eigenvalues and eigenspaces of the Jacobi operator $R_W = R(\cdot, W)W$ for a singular unit tangent vector W .

1. If W is an \mathfrak{A} -principal singular unit tangent vector with respect to $A \in \mathfrak{A}$, then the eigenvalues of R_W are 0 and 2 and the corresponding eigenspaces are $\mathbb{R}W \oplus J(V(A) \ominus \mathbb{R}W)$ and $(V(A) \ominus \mathbb{R}W) \oplus \mathbb{R}JW$, respectively.
2. If W is an \mathfrak{A} -isotropic singular unit tangent vector with respect to $A \in \mathfrak{A}$ and $X, Y \in V(A)$, then the eigenvalues of R_W are 0, 1 and 4 and the corresponding eigenspaces are $\mathbb{R}W \oplus \mathbb{C}(JX + Y)$, $T_{[z]}Q^m \ominus (\mathbb{C}X \oplus \mathbb{C}Y)$ and $\mathbb{R}JW$, respectively.

3. SOME GENERAL EQUATIONS

Let M be a real hypersurface in Q^m and denote by (ϕ, ξ, η, g) the induced almost contact metric structure. Note that $\xi = -JN$, where N is a (local) unit normal vector field of M and η the corresponding 1-form defined by $\eta(X) = g(\xi, X)$ for any tangent vector field X on M . The tangent bundle TM of M splits orthogonally into $TM = \mathcal{C} \oplus \mathbb{R}\xi$, where $\mathcal{C} = \ker(\eta)$ is the maximal complex subbundle of TM . The structure tensor field ϕ restricted to \mathcal{C} coincides with the complex structure J restricted to \mathcal{C} , and $\phi\xi = 0$.

At each point $z \in M$ we define a maximal \mathfrak{A} -invariant subspace of T_zM , $z \in M$ as follows:

$$\mathcal{Q}_z = \{X \in T_zM \mid AX \in T_zM \text{ for all } A \in \mathfrak{A}_z\}.$$

Then we want to introduce an important lemma which will be used in the proof of our main Theorem in the introduction.

Lemma 3.1. ([16] and [18]) *For each $z \in M$ we have*

- (i) *If N_z is \mathfrak{A} -principal, then $\mathcal{Q}_z = \mathcal{C}_z$.*
- (ii) *If N_z is not \mathfrak{A} -principal, there exist a conjugation $A \in \mathfrak{A}$ and orthonormal vectors $X, Y \in V(A)$ such that $N_z = \cos(t)X + \sin(t)JY$ for some $t \in (0, \pi/4]$. Then we have $\mathcal{Q}_z = \mathcal{C}_z \ominus \mathbb{C}(JX + Y)$.*

We now assume that M is a Hopf hypersurface. Then we have

$$S\xi = \alpha\xi,$$

where S denotes the shape operator of the real hypersurfaces M with the smooth function $\alpha = g(S\xi, \xi)$ on M . When we consider the transform JX by the Kähler structure J on Q^m for any vector field X on M in Q^m , we may put

$$JX = \phi X + \eta(X)N$$

for a unit normal N to M . Then we now consider the Codazzi equation

$$\begin{aligned} g((\nabla_X S)Y - (\nabla_Y S)X, Z) &= \eta(X)g(\phi Y, Z) - \eta(Y)g(\phi X, Z) - 2\eta(Z)g(\phi X, Y) \\ &\quad + g(X, AN)g(AY, Z) - g(Y, AN)g(AX, Z) \\ &\quad + g(X, A\xi)g(JAY, Z) - g(Y, A\xi)g(JAX, Z). \end{aligned} \quad (3.1)$$

Putting $Z = \xi$ in (3.1) we get

$$\begin{aligned} g((\nabla_X S)Y - (\nabla_Y S)X, \xi) &= -2g(\phi X, Y) \\ &\quad + g(X, AN)g(Y, A\xi) - g(Y, AN)g(X, A\xi) \\ &\quad - g(X, A\xi)g(JY, A\xi) + g(Y, A\xi)g(JX, A\xi). \end{aligned}$$

On the other hand, we have

$$\begin{aligned} &g((\nabla_X S)Y - (\nabla_Y S)X, \xi) \\ &= g((\nabla_X S)\xi, Y) - g((\nabla_Y S)\xi, X) \\ &= (X\alpha)\eta(Y) - (Y\alpha)\eta(X) + \alpha g((S\phi + \phi S)X, Y) - 2g(S\phi SX, Y). \end{aligned}$$

Comparing the previous two equations and putting $X = \xi$ yields

$$Y\alpha = (\xi\alpha)\eta(Y) - 2g(\xi, AN)g(Y, A\xi) + 2g(Y, AN)g(\xi, A\xi).$$

Reinserting this into the previous equation yields

$$\begin{aligned}
& g((\nabla_X S)Y - (\nabla_Y S)X, \xi) \\
&= -2g(\xi, AN)g(X, A\xi)\eta(Y) + 2g(X, AN)g(\xi, A\xi)\eta(Y) \\
&\quad + 2g(\xi, AN)g(Y, A\xi)\eta(X) - 2g(Y, AN)g(\xi, A\xi)\eta(X) \\
&\quad + \alpha g((\phi S + S\phi)X, Y) - 2g(S\phi SX, Y).
\end{aligned}$$

Altogether this implies

$$\begin{aligned}
0 &= 2g(S\phi SX, Y) - \alpha g((\phi S + S\phi)X, Y) - 2g(\phi X, Y) \\
&\quad + g(X, AN)g(Y, A\xi) - g(Y, AN)g(X, A\xi) \\
&\quad - g(X, A\xi)g(JY, A\xi) + g(Y, A\xi)g(JX, A\xi) \\
&\quad + 2g(\xi, AN)g(X, A\xi)\eta(Y) - 2g(X, AN)g(\xi, A\xi)\eta(Y) \\
&\quad - 2g(\xi, AN)g(Y, A\xi)\eta(X) + 2g(Y, AN)g(\xi, A\xi)\eta(X).
\end{aligned} \tag{3.2}$$

At each point $z \in M$ we can choose $A \in \mathfrak{A}_z$ such that

$$N = \cos(t)Z_1 + \sin(t)JZ_2$$

for some orthonormal vectors $Z_1, Z_2 \in V(A)$ and $0 \leq t \leq \frac{\pi}{4}$ (see Proposition 3 in [10]). Note that t is a function on M . First of all, since $\xi = -JN$, we have

$$\begin{aligned}
N &= \cos(t)Z_1 + \sin(t)JZ_2, \\
AN &= \cos(t)Z_1 - \sin(t)JZ_2, \\
\xi &= \sin(t)Z_2 - \cos(t)JZ_1, \\
A\xi &= \sin(t)Z_2 + \cos(t)JZ_1.
\end{aligned} \tag{3.3}$$

This implies $g(\xi, AN) = 0$ and hence

$$\begin{aligned}
0 &= 2g(S\phi SX, Y) - \alpha g((\phi S + S\phi)X, Y) - 2g(\phi X, Y) \\
&\quad + g(X, AN)g(Y, A\xi) - g(Y, AN)g(X, A\xi) \\
&\quad - g(X, A\xi)g(JY, A\xi) + g(Y, A\xi)g(JX, A\xi) \\
&\quad - 2g(X, AN)g(\xi, A\xi)\eta(Y) + 2g(Y, AN)g(\xi, A\xi)\eta(X).
\end{aligned} \tag{3.4}$$

4. RICCI COMMUTING AND A KEY LEMMA

By the equation of Gauss, the curvature tensor $R(X, Y)Z$ for a real hypersurface M in Q^m induced from the curvature tensor \bar{R} of Q^m can be described in terms of the complex structure J and the complex conjugation $A \in \mathfrak{A}$ as follows:

$$\begin{aligned}
R(X, Y)Z &= g(Y, Z)X - g(X, Z)Y + g(\phi Y, Z)\phi X - g(\phi X, Z)\phi Y - 2g(\phi X, Y)\phi Z \\
&\quad + g(AY, Z)AX - g(AX, Z)AY + g(JAY, Z)JAX - g(JAX, Z)JAY \\
&\quad + g(SY, Z)SX - g(SX, Z)SY
\end{aligned}$$

for any $X, Y, Z \in T_z M$, $z \in M$.

Now let us put

$$AX = BX + \rho(X)N,$$

for any vector field $X \in T_z Q^m$, $z \in M$, $\rho(X) = g(AX, N)$, where BX and $\rho(X)N$ respectively denote the tangential and normal component of the vector field AX . Then $A\xi = B\xi + \rho(\xi)N$ and $\rho(\xi) = g(A\xi, N) = 0$. Then it follows that

$$\begin{aligned} AN &= AJ\xi = JA\xi = -J(B\xi + \rho(\xi)N) \\ &= -(\phi B\xi + \eta(B\xi)N). \end{aligned}$$

The equation gives $g(AN, N) = -\eta(B\xi)$ and $g(AN, \xi) = 0$. From this, together with the definition of the Ricci tensor, we have

$$\begin{aligned} Ric(X) &= (2m-1)X - 3\eta(X)\xi - g(AN, N)AX + g(AX, N)AN \\ &\quad + \eta(AX)A\xi + (TrS)SX - S^2X. \end{aligned}$$

Then, summing up with the above formulas, it can be rearranged as follows:

$$\begin{aligned} Ric(X) &= (2m-1)X - 3\eta(X)\xi + \eta(B\xi)\{BX + \rho(X)N\} \\ &\quad + \rho(X)\{-\phi B\xi - \eta(B\xi)N\} + \eta(BX)B\xi + (TrS)SX - S^2X \end{aligned}$$

From this, together with the assumption of Ricci commuting, that is, $\phi \cdot Ric(X) = Ric \cdot \phi X$, it follows that

$$\begin{aligned} &(2m-1)\phi X + \eta(B\xi)\phi BX - \rho(X)\phi^2 B\xi \\ &\quad + \eta(BX)\phi B\xi + (TrS)\phi SX - \phi S^2X \\ &= (2m-1)\phi X + \eta(B\xi)B\phi X - \rho(\phi X)\phi B\xi \\ &\quad + \eta(B\phi X)B\xi + (TrS)S\phi X - S^2\phi X. \end{aligned} \tag{4.1}$$

Here we want to use the following formulas

$$\begin{aligned} \eta(BX) &= g(A\xi, X), \\ \eta(B\phi X) &= g(A\xi, \phi X) = g(A\xi, JX - \eta(X)N) = g(AJ\xi, X) \\ &= g(AN, X) = \rho(X), \\ \rho(\phi X) &= g(A\phi X, N) = g(AJ\phi X, \xi) = g(J\phi X, A\xi) \\ &= g(\phi^2 X + \eta(\phi X)N, A\xi) = -g(X, A\xi) + \eta(X)g(\xi, A\xi), \\ \rho(X) &= \eta(B\phi X). \end{aligned}$$

Summing up these formulas into (4.1), we have

$$\begin{aligned} &\eta(B\xi)\phi BX - \eta(B\phi X)\eta(B\xi)\xi + (TrS)\phi SX - \phi S^2X \\ &= \eta(B\xi)B\phi X - \eta(X)\eta(B\xi)\phi B\xi + (TrS)S\phi X - S^2\phi X. \end{aligned} \tag{4.2}$$

Then, by taking inner product of (4.2) with ξ and using that M is Hopf, it follows that

$$\eta(B\xi)\phi B\xi = 0. \tag{4.3}$$

Then the formula (4.2) becomes

$$\eta(B\xi)(\phi B - B\phi)X + (TrS)(\phi S - S\phi)X - (\phi S^2 - S^2\phi)X = 0. \tag{4.4}$$

Here we want to give a remark as follows:

Remark 4.1. *Let M be a real hypersurface over a totally geodesic $\mathbb{C}P^k \subset Q^{2k}$, $m = 2k$. Then in papers due to [16] and [18] the structure tensor commutes with the shape operator, that is, $S\phi = \phi S$. Moreover, the unit normal vector field N becomes \mathfrak{A} -isotropic. This gives $\eta(B\xi) = g(A\xi, \xi) = 0$. So it naturally satisfies the formula (4.2), that is, Ricci commuting.*

On the other hand, from (4.3) we assert an important lemma as follows:

Lemma 4.2. *Let M be a real hypersurface in Q^m , $m \geq 3$, with commuting Ricci tensor. Then the unit normal vector field N becomes singular, that is, N is \mathfrak{A} -isotropic or \mathfrak{A} -principal.*

Proof. From (4.3) we get

$$\eta(B\xi) = 0 \quad \text{or} \quad \phi B\xi = 0.$$

The first case gives that $\eta(B\xi) = g(A\xi, \xi) = \cos 2t = 0$, that is, $t = \frac{\pi}{4}$. This implies that the unit normal N becomes $N = \frac{X+JY}{\sqrt{2}}$, which means that N is \mathfrak{A} -isotropic.

The second case gives that

$$\rho(X) = g(AX, N) = \eta(B\phi X) = -g(X, \phi B\xi) = 0,$$

which means that $AX \in T_z M$ for any $A \in \mathfrak{A}$, $X \in T_z M$, $z \in M$. This implies $\mathcal{Q}_z = \mathcal{C}_z$, $z \in M$, and N is \mathfrak{A} -principal, that is, $AN = N$. \square

In order to prove our main theorem in the introduction, by virtue of Lemma 4.2, we can divide into two classes of hypersurfaces in Q^m with the unit normal N is \mathfrak{A} -principal or \mathfrak{A} -isotropic. When M is with \mathfrak{A} -isotropic, in section 5 we will give its proof in detail and in section 6 we will give the remainder proof for the case that M has a \mathfrak{A} -principal normal vector field.

5. PROOF OF MAIN THEOREM WITH \mathfrak{A} -ISOTROPIC

In this section we want to prove our Main Theorem for real hypersurfaces M in Q^m with commuting Ricci tensor when the unit normal vector field becomes \mathfrak{A} -isotropic.

Since we assumed that the unit normal N is \mathfrak{A} -isotropic, by the definition in section 3 we know that $t = \frac{\pi}{4}$. Then by the expression of the \mathfrak{A} -isotropic unit normal vector field, (3.3) gives $N = \frac{1}{\sqrt{2}}Z_1 + \frac{1}{\sqrt{2}}JZ_2$. This implies that $g(A\xi, \xi) = 0$. Since the unit normal N is \mathfrak{A} -isotropic, we know that $g(\xi, A\xi) = 0$. Moreover, by (3.4) and using an anti-commuting property $AJ = -JA$ between the complex conjugation A and the Kähler structure J , we proved the following (see also Lemma 4.2 in [16])

Lemma 5.1. *Let M be a Hopf hypersurface in Q^m with (local) unit normal vector field N . For each point in $z \in M$ we choose $A \in \mathfrak{A}_z$ such that $N_z = \cos(t)Z_1 + \sin(t)JZ_2$ holds for some orthonormal vectors $Z_1, Z_2 \in V(A)$ and $0 \leq t \leq \frac{\pi}{4}$. Then*

$$\begin{aligned} 0 &= 2g(S\phi SX, Y) - \alpha g((\phi S + S\phi)X, Y) - 2g(\phi X, Y) \\ &\quad + 2g(X, AN)g(Y, A\xi) - 2g(Y, AN)g(X, A\xi) \\ &\quad + 2g(\xi, A\xi)\{g(Y, AN)\eta(X) - g(X, AN)\eta(Y)\} \end{aligned}$$

holds for all vector fields X, Y on M .

Then by virtue of \mathfrak{A} -isotropic unit normal, Lemma 5.1 becomes the following

$$2S\phi SX = \alpha(S\phi + \phi S)X + 2\phi X - 2g(X, AN)A\xi + 2g(X, A\xi)AN. \quad (5.1)$$

Now let us consider the distribution \mathcal{Q}^\perp , which is an orthogonal complement of the maximal \mathfrak{A} -invariant subspace \mathcal{Q} in the complex subbundle \mathcal{C} of $T_z M$, $z \in M$ in Q^m . Then by Lemma 3.1 in section 3, the orthogonal complement $\mathcal{Q}^\perp = \mathcal{C} \ominus \mathcal{Q}$ becomes $\mathcal{C} \ominus \mathcal{Q} = \text{Span} [AN, A\xi]$. From the assumption of $S\phi = \phi S$ on the distribution \mathcal{Q}^\perp it can be easily checked that the distribution \mathcal{Q}^\perp is invariant by the shape operator S . Then (5.1) gives the following for $SAN = \lambda AN$

$$\begin{aligned} (2\lambda - \alpha)S\phi AN &= (\alpha\lambda + 2)\phi AN - 2A\xi \\ &= (\alpha\lambda + 2)\phi AN - 2\phi AN \\ &= \alpha\lambda\phi AN. \end{aligned}$$

Then $A\xi = \phi AN$ gives the following

$$SA\xi = \frac{\alpha\lambda}{2\lambda - \alpha}A\xi. \quad (5.2)$$

Then from the assumption $S\phi = \phi S$ on $\mathcal{Q}^\perp = \mathcal{C} \ominus \mathcal{Q}$ it follows that $\lambda = \frac{\alpha\lambda}{2\lambda - \alpha}$ gives

$$\lambda = 0 \text{ or } \lambda = \alpha. \quad (5.3)$$

On the other hand, on the distribution \mathcal{Q} we know that $AX \in T_z M$, $z \in M$, because $AN \in \mathcal{Q}$. So (5.1), together with the fact that $g(X, A\xi) = 0$ and $g(X, AN) = 0$ for any $X \in \mathcal{Q}$, imply that

$$2S\phi SX = \alpha(S\phi + \phi S)X + 2\phi X. \quad (5.4)$$

Then we can take an orthonormal basis $X_1, \dots, X_{2(m-2)} \in \mathcal{Q}$ such that $AX_i = \lambda_i X_i$ for $i = 1, \dots, m-2$. Then by (5.1) we know that

$$S\phi X_i = \frac{\alpha\lambda_i + 2}{2\lambda_i - \alpha}\phi X_i.$$

Accordingly, by (5.3) the shape operator S can be expressed in such a way that

$$S = \begin{bmatrix} \alpha & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0(\alpha) & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0(\alpha) & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \lambda_1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \lambda_{m-2} & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & \mu_1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & \mu_{m-2} \end{bmatrix}$$

From the equation (4.4), together with $\eta(B\xi) = g(A\xi, \xi) = 0$, we have that

$$h(\phi S - S\phi)X = (\phi S^2 - S^2\phi)X, \quad (5.5)$$

where $h = \text{Tr} S$ denotes the trace of the shape operator of M in Q^m .

Now let us consider the Ricci commuting with \mathfrak{A} -isotropic normal for $SX = \lambda X$, $X \in \mathcal{C}$. Then from (5.5) it follows that for $SX = \lambda X$, $X \in \mathcal{C}$

$$(\lambda - \mu)\{h - (\lambda + \mu)\}\phi X = 0, \quad (5.6)$$

where we have used $S\phi X = \mu\phi X$ for $\mu = \frac{\alpha\lambda+2}{2\lambda-\alpha}$, $X \in \mathcal{Q}$ and $\mu = \frac{\alpha\lambda}{2\lambda-\alpha}$, $X \in \mathcal{Q}^\perp$ respectively. Then (5.6) gives that

$$\lambda = \mu \quad \text{or} \quad h = \lambda + \mu. \quad (5.7)$$

On the other hand, we consider the following for $SX = \lambda X$, $X \in \mathcal{C}$. Then (5.5) gives

$$h\lambda\phi X - hS\phi X = \lambda^2\phi X - S^2\phi X. \quad (5.8)$$

Here we decompose $X \in \mathcal{C} = \mathcal{Q} \oplus \mathcal{Q}^\perp$ in such a way that

$$X = Y + Z,$$

where $Y \in \mathcal{Q}$ and $Z \in \mathcal{Q}^\perp$. Then $SX = \lambda X = \lambda Y + \lambda Z$ gives the following

$$SY = \lambda Y \quad \text{and} \quad SZ = \lambda Z,$$

because the distribution \mathcal{Q} and \mathcal{Q}^\perp are invariant by the shape operator. Then by using the matrix representation of the shape operator the formula (5.8) gives the following decomposition

$$h\lambda\phi Y - h\left(\frac{\alpha\lambda+2}{2\lambda-\alpha}\right)\phi Y = \lambda^2\phi Y - \left(\frac{\alpha\lambda+2}{2\lambda-\alpha}\right)^2\phi Y, \quad Y \in \mathcal{Q}, \quad (5.9)$$

$$h\lambda\phi Z - h\left(\frac{\alpha\lambda}{2\lambda-\alpha}\right)\phi Z = \lambda^2\phi Z - \left(\frac{\alpha\lambda}{2\lambda-\alpha}\right)^2\phi Z, \quad Z \in \mathcal{Q}^\perp. \quad (5.10)$$

By taking inner products of (5.9) and (5.10) with the vector fields ϕY and ϕZ respectively, we have

$$\lambda^2 - h\lambda + \frac{\alpha\lambda+2}{2\lambda-\alpha}\{h - \frac{\alpha\lambda+2}{2\lambda-\alpha}\} = 0, \quad (5.11)$$

$$\lambda^2 - h\lambda + \frac{\alpha\lambda}{2\lambda-\alpha}\{h - \frac{\alpha\lambda}{2\lambda-\alpha}\} = 0. \quad (5.12)$$

Then subtracting (5.12) from (5.11) gives

$$h = \frac{2\alpha\lambda+2}{2\lambda-\alpha}. \quad (5.13)$$

Now from (5.7) let us consider the following two cases:

Case I. $\lambda = \mu$

From the matrix representation of the shape operator, $\lambda = \frac{\alpha\lambda+2}{2\lambda-\alpha}$ gives that

$$\lambda^2 - \alpha\lambda - 1 = 0.$$

Since the discriminant $D = \alpha^2 + 4 > 0$, we have two distinct solutions $\lambda = \cot r$ and $\mu = -\tan r$ with the multiplicities $(m-2)$ and $(m-2)$ respectively.

That is, the shape operator S can be expressed in such a way that

$$S = \begin{bmatrix} \alpha & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0(\alpha) & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0(\alpha) & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cot r & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \cot r & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & -\tan r & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & -\tan r \end{bmatrix}$$

This means that the shape operator S commutes with the structure tensor ϕ , that is, $S \cdot \phi = \phi \cdot S$. Then by Theorem C, $m = 2k$, and M is locally congruent to an open part of a tube around a totally geodesic $\mathbb{C}P^k$ in Q^{2k} .

Case II. $\lambda \neq \mu$

Now we only consider $\lambda \neq \mu$ on the distribution \mathcal{Q} . Since on the distribution \mathcal{Q}^\perp we have assumed that $S\phi = \phi S$, so it follows that $\lambda = \frac{\alpha\lambda}{2\lambda - \alpha}$. This gives $\lambda = 0$ or $\lambda = \alpha$ on the distribution \mathcal{Q}^\perp . Moreover, by the Ricci commuting, we have the following from (5.3), together with (5.6), (5.7) and (5.13)

$$\begin{aligned} h = \lambda + \mu &= \lambda + \frac{\alpha\lambda + 2}{2\lambda - \alpha} \\ &= \frac{2\alpha\lambda + 2}{2\lambda - \alpha}. \end{aligned} \tag{5.14}$$

This gives $\lambda = 0$ or $\lambda = \alpha$. Then we can divide into two subcases as follows:

Subcase 2.1. $\lambda = 0$.

Then we can arrange the matrix of the shape operator such that

$$S = \begin{bmatrix} \alpha & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0(\alpha) & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0(\alpha) & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & -\frac{2}{\alpha} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & -\frac{2}{\alpha} \end{bmatrix}$$

In this case the formula $h = \lambda + \mu$ and the notion of trace h of the shape operator S gives

$$\begin{aligned} h &= 0 - \frac{2}{\alpha} \\ &= \alpha + (m-2)\left(-\frac{2}{\alpha}\right). \end{aligned}$$

Then it gives that $\alpha^2 = 2(m-3)$, that is, $\alpha = \sqrt{2(2m-3)}$.

Now let us consider another case that $h = 3\alpha + (m-2)(-\frac{2}{\alpha})$. Then, from this, together with $h = \lambda + \mu = -\frac{2}{\alpha}$, we know $\alpha^2 = \frac{2}{3}(m-3)$. Then $\alpha = \sqrt{\frac{2}{3}(m-3)}$.

Subcase 2.2. $\lambda = \alpha$.

In this subcase, the expression of the shape operator beomes

$$S = \begin{bmatrix} \alpha & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \alpha(0) & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & \alpha(0) & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \alpha & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \alpha & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & \frac{\alpha^2+2}{\alpha} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & \frac{\alpha^2+2}{\alpha} \end{bmatrix}$$

In this case also the formula $h = \lambda + \mu$ and the notion of trace h of the shape operator S gives

$$\begin{aligned} h &= \alpha + \frac{\alpha^2 + 2}{\alpha} \\ &= (m+1)\alpha + (m-2)\frac{\alpha^2 + 2}{\alpha}. \end{aligned}$$

Then it implies that $(2m-3)\alpha^2 = -2m+4$, which gives us a contradiction for $m \geq 3$.

Next we consider another case that the trace becomes $h = (m-1)\alpha + (m-2)\frac{\alpha^2+2}{\alpha}$ in the above expression. Then $h = \lambda + \mu = \alpha + \frac{\alpha^2+2}{\alpha}$ gives

$$(2m-5)\alpha^2 + 2(m-3) = 0,$$

which also implies a contradiction for $m \geq 4$.

Summing up the above discussions, we assert the following

Theorem 1. *Let M be a real hypersurface in complex quadric Q^m , $m \geq 4$, with commuting Ricci tensor and \mathfrak{A} -isotropic normal. If the shape operator commutes with the structure tensor on the distribution \mathcal{Q}^\perp , then M is locally congruent to a tube of radius r over a totally geodesic $\mathbb{C}P^k$, $m = 2k$, in Q^{2k} or M has 3 distinct constant principal curvatures given by*

$$\begin{aligned} \alpha &= \sqrt{2(m-3)}, \gamma = 0, \lambda = 0, \text{ and } \mu = -\frac{2}{\sqrt{2(m-3)}} \quad \text{or} \\ \alpha &= \sqrt{\frac{2}{3}(m-3)}, \gamma = 0, \lambda = 0, \text{ and } \mu = -\frac{\sqrt{6}}{\sqrt{m-3}} \end{aligned}$$

with corresponding principal curvature spaces

$$T_\alpha = [\xi], T_\gamma = [A\xi, AN], \phi(T_\lambda) = T_\mu, \dim T_\lambda = \dim T_\mu = m-2.$$

6. PROOF OF MAIN THEOREM WITH \mathfrak{A} -PRINCIPAL

In this section we want to prove our Main Theorem for real hypersurfaces with commuting Ricci tensor and \mathfrak{A} -principal unit normal vector field.

From the basic formulas for the real structure A and the Kähler structure J we have the following

$$JAX = J\{BX + \rho(X)N\} = \phi BX + \eta(BX)N - \rho(X)\xi,$$

$$AJX = A\{\phi X + \eta(X)N\} = B\phi X + \rho(\phi X)N - \eta(X)\phi B\xi - \eta(X)\eta(B\xi)N.$$

From this, the anti-commuting structure $AJ = -JA$ gives the following

$$\phi BX + \eta(BX)N - \rho(X)\xi = -B\phi X - \rho(\phi X)N + \eta(X)\phi B\xi + \eta(X)\eta(B\xi)N. \quad (6.1)$$

Then comparing the tangential and normal component of (6.1) gives the following respectively

$$\eta(BX) = -\rho(\phi X) + \eta(X)\eta(B\xi), \quad (6.2)$$

and

$$\phi BX = -B\phi X + \rho(X)\xi + \eta(X)\phi B\xi. \quad (6.3)$$

Since N is \mathfrak{A} -principal, that is, $AN = N$, we know that $B\xi = -\xi$, and $\phi BX = -B\phi X$. Then (6.3) gives

$$-2\phi BX + (TrS)(\phi S - S\phi)X - (\phi S^2 - S^2\phi)X = 0, \quad (6.4)$$

where we have used $\rho(X) = g(AX, N) = 0$. When N is \mathfrak{A} -principal, on the distribution $\mathcal{C} = \mathcal{Q}$ we have

$$2S\phi S - \alpha(\phi S + S\phi) = 2\phi. \quad (6.5)$$

So if we put $SX = \lambda X$ in (6.5), we have

$$S\phi X = \mu\phi X = \frac{\alpha\lambda + 2}{2\lambda - \alpha}\phi X. \quad (6.6)$$

Then from (6.3) and (6.5) it follows that

$$-2\phi BX + (\lambda - \mu)\{h - (\lambda + \mu)\}\phi X = 0. \quad (6.7)$$

It is well known that the tangent space $T_z Q^m$ of the complex quadric Q^m is decomposed as

$$T_z Q^m = V(A) \oplus JV(A),$$

where $V(A) = \{X \in T_z Q^m | AX = X\}$ and $JV(A) = \{X \in T_z Q^m | AX = -X\}$. So $SX = \lambda X$ for $X \in \mathcal{C}$ the vector field X can be decomposed as follows:

$$X = Y + Z, \quad Y \in V(A), \quad Z \in JV(A),$$

where $AY = BY = Y$ and $AZ = BZ = -Z$. So it follows that $BX = AX = AY + AZ = Y - Z$. Then $\phi BX = \phi Y - \phi Z$. From this, together with (6.7), it follows that

$$-2(\phi Y - \phi Z) + (\lambda - \mu)\{h - (\lambda + \mu)\}(\phi Y + \phi Z) = 0.$$

Then by taking inner products with ϕY and ϕZ respectively, we get

$$(\lambda - \mu)\{h - (\lambda + \mu)\} - 2 = 0 \quad (6.8)$$

and

$$(\lambda - \mu)\{h - (\lambda + \mu)\} + 2 = 0. \quad (6.9)$$

This gives a contradiction. Accordingly, we conclude that real hypersurfaces in Q^m with commuting Ricci tensor and \mathfrak{A} -principal normal do not exist.

Summing up the above discussions, we assert the following

Theorem 2. *There do not exist any real hypersurface in complex quadric Q^m , $m \geq 4$, with commuting Ricci tensor and \mathfrak{A} -principal normal vector field.*

From Theorems 1 and 2, together with Lemma 4.2, we give a complete proof of our Main Theorem in the introduction.

Remark 6.1. *In this paper we have asserted that a tube over a totally geodesic $\mathbb{C}P^k$ in Q^m , $m = 2k$, mentioned in our Main Theorem is Ricci commuting, that is, $\text{Ric} \cdot \phi = \phi \cdot \text{Ric}$. But related to the notion of Ricci parallel, in the paper [18] we asserted that a tube over $\mathbb{C}P^k$ never has a parallel Ricci tensor, that is, the Ricci tensor does not satisfy $\nabla S = 0$.*

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